Diagonalization of Difference Operators and System Matrices in the Method of Lines

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Abstract— **In the method of lines, the eigensolutions for the difference operator for the electric field can be calculated from the corresponding eigensolutions for the magnetic field. For an arbitrary discretization of a homogeneous layer it is proved that this is achieved by a simple matrix multiplication, which yields numerical advantages, especially in the cases of nonequidistant discretization or absorbing boundaries. Secondly, a transformation to principal axes of the system matrices for multilayered planar structures is given which enables an easier transfer of the field components from one layer interface to the other. The result corresponds to that of the immittance approach in the spectral-domain method.**

I. Introduction

THE method of lines has been successfully applied to a variety of planar microwave [1] and optical [2] waveguide structures. One of its advantages is the easy formulation, which is mainly achieved by suitably taking advantage of both the planar structure of the waveguides and the properties of the matrices encountered [1], [3].

In this letter, the diagonalization of two particular matrices used in the method of lines is investigated. First, we prove in a general form that the eigensolutions of the second-order difference operator for the electric field [1], [4] can be calculated from the eigensolutions of the corresponding operator for the magnetic field. The transformed first order difference operators are also derived from the given eigensolutions.

For the analysis of structures comprising a large number of dielectric layers, e.g., four or more as in Fig. 1, the transfer relation from one interface plane to another [1] is improved. The system matrices for multilayered planar structures are transformed to principal axes in order to ease the transfer of the field components from one layer interface to the other. The result corresponds to that of the immittance approach [5] in the spectral-domain method. In a recent paper [6], this was tried in a similar way. However, the transfer matrix approach given in [1] is not taken into account there, but an old formulation [7] is cited. Moreover, the formulas given in [6] are not directly applicable to more than three layers. Our approach does not suffer from these disadvantages and is adapted very well to the state of the art matrix analysis in the method of lines. The analysis is also applied to three-dimensional problems.

The following investigations are carried out on the basis of the comprehensive book article about the method of lines [1], especially Sections 2.3–2.4, 2.7, 3.4 and Appendixes A and B.

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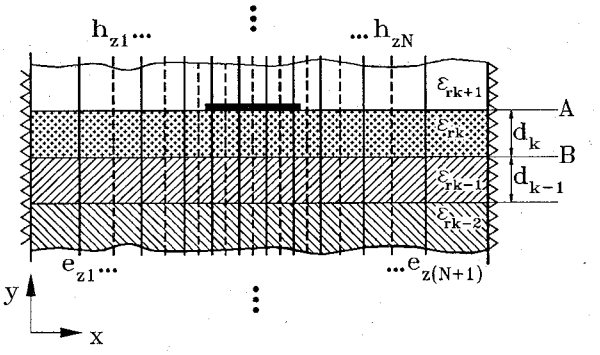

Fig. 1. Nonequidistant discretization of a multilayered planar structure with absorbing lateral boundaries.

II. Relations Between the Diagonalized Difference Operators

The analysis of this section is valid for arbitrary lateral boundary conditions, including the present implementation of absorbing walls [4], and also for nonequidistant discretization. We discretize the field components e_z and h_z as illustrated in Fig. 1 and their first derivatives with respect to x according to

$$\frac{\partial e_z}{\partial \bar{x}} \rightarrow \boldsymbol{D}_e \boldsymbol{E}_z \quad ; \quad \frac{\partial h_z}{\partial \bar{x}} \rightarrow \boldsymbol{D}_h \boldsymbol{H}_z \tag{1}$$

using coordinates normalized by the wavenumber, e.g., $\bar{x} = k_0 x$. The difference operators $\boldsymbol{D}_{e,h}$ are completely general matrices and no band or other structure is necessary. We use shifted line systems for e_z and h_z as in [1, Section 2.2].

For a homogeneous layer the second order difference operators $\boldsymbol{P}_{e,h}$ are the negative products of two different first order operators and their eigenvalues $\boldsymbol{\lambda}^2_{e,h}$ are defined as follows:

$$\boldsymbol{P}_h \boldsymbol{T}_h = -\boldsymbol{D}_e \boldsymbol{D}_h \boldsymbol{T}_h = \boldsymbol{T}_h \boldsymbol{\lambda}^2_h \tag{2}$$

$$\boldsymbol{P}_e \boldsymbol{T}_e = -\boldsymbol{D}_h \boldsymbol{D}_e \boldsymbol{T}_e = \boldsymbol{T}_e \boldsymbol{\lambda}^2_e. \tag{3}$$

Assuming that $\boldsymbol{\lambda}^2_h$ and the eigenvector matrix $\boldsymbol{T}_h$ for the magnetic difference operator $\boldsymbol{P}_h$ are given, the electric eigensolutions $\boldsymbol{T}_e$ and $\boldsymbol{\lambda}^2_e$ can be computed [8],[9] using

$$-\boldsymbol{D}_h \boldsymbol{D}_e (\boldsymbol{D}_h \boldsymbol{T}_h) = (\boldsymbol{D}_h \boldsymbol{T}_h) \boldsymbol{\lambda}^2_h. \tag{4}$$

1) The first case we consider is that both lateral boundaries lie on e_z-lines as in Fig. 1, which implies that the dimensions of $\boldsymbol{P}_h$ and $\boldsymbol{P}_e$ are $N \times N$ and $(N+1) \times (N+1)$, respectively. We obtain

$$\boldsymbol{\lambda}^2_e = \begin{bmatrix} 0 & \\ & \boldsymbol{\lambda}^2_h \end{bmatrix}; \; \boldsymbol{T}_e = [\boldsymbol{t}_0 \vdots \underbrace{\boldsymbol{D}_h \boldsymbol{T}_h \cdot \boldsymbol{\mu}_h}_{\boldsymbol{T}^r_e}], \tag{5}$$

where the vector $\boldsymbol{t}_0$ is a nontrivial solution of $\boldsymbol{D}_e \boldsymbol{t}_0 = \boldsymbol{o}$, which

is unique but for a scaling factor. The arbitrary diagonal matrix $\boldsymbol{\mu}_h$ is chosen to be $\boldsymbol{\lambda}_h^{-1}$ for normalization.

2) In the case of equal dimensions of $\boldsymbol{P}_e$ and $\boldsymbol{P}_h$ (i.e., with the lateral boundaries on different line systems), the electric eigensolutions are given by

$$\boldsymbol{\lambda}_e^2 = \boldsymbol{\lambda}_h^2 \;;\; \boldsymbol{T}_e = \boldsymbol{D}_h \boldsymbol{T}_h \boldsymbol{\lambda}_h^{-1} \tag{6}$$

in analogy to (5).

3) If both lateral boundaries lie on h_z-lines, the dimensions of the difference operators are exchanged with each other. Hence, (4)–(5) are valid again, if we swap the subscripts e and h. Using $\boldsymbol{\mu}_e = -\boldsymbol{\lambda}_e^{-1}$ we obtain

$$\begin{bmatrix} 0 & \\ & \boldsymbol{\lambda}_e^2 \end{bmatrix} = \boldsymbol{\lambda}_h^2 \;;\; \left[\boldsymbol{o} \,\vdots\, \boldsymbol{T}_e\right] = \boldsymbol{D}_h \boldsymbol{T}_h \boldsymbol{\lambda}_h^{-1}. \tag{7}$$

As can be seen from (5)–(7), the transformation matrix $\boldsymbol{T}_e$ can be calculated from $\boldsymbol{T}_h$ by a *simple matrix multiplication.* This saves computing time, especially when the eigensolutions must be determined numerically, as in the cases of nonequidistant discretization or absorbing boundaries.

The transforms of the two first-order difference operators are given by

$$\boldsymbol{\delta}_e = \boldsymbol{T}_h^{-1} \boldsymbol{D}_e \boldsymbol{T}_e \;;\; \boldsymbol{\delta}_h = \boldsymbol{T}_e^{-1} \boldsymbol{D}_h \boldsymbol{T}_h \tag{8}$$

in the case of boundaries on e_z-lines on both sides. Using (5) we obtain

$$\boldsymbol{\delta}_h = \boldsymbol{T}_e^{-1} \cdot \boldsymbol{T}_e^r \boldsymbol{\lambda}_h = \begin{bmatrix} \boldsymbol{o}^t \\ \cdots \\ \boldsymbol{\lambda}_h \end{bmatrix} \tag{9}$$

and with (5) and (2),

$$\begin{aligned} \boldsymbol{\delta}_e &= \boldsymbol{T}_h^{-1} \boldsymbol{D}_e \left[\boldsymbol{t}_0 \,\vdots\, \boldsymbol{D}_h \boldsymbol{T}_h \boldsymbol{\lambda}_h^{-1}\right] \\ &= \left[\boldsymbol{T}_h^{-1} \boldsymbol{D}_e \boldsymbol{t}_0 \,\vdots\, \boldsymbol{T}_h^{-1} \boldsymbol{D}_e \boldsymbol{D}_h \boldsymbol{T}_h \boldsymbol{\lambda}_h^{-1}\right] \\ \boldsymbol{\delta}_e &= -\left[\boldsymbol{o} \,\vdots\, \boldsymbol{\lambda}_h\right] = -\boldsymbol{\delta}_h^t. \end{aligned} \tag{10}$$

The other boundary combinations yield corresponding results.

For nonequidistant discretization as in [1, Section 2.7], we use the normalized difference operators $\bar{\boldsymbol{D}}$ and $-\bar{\boldsymbol{D}}^t$ instead of $\boldsymbol{D}_e$ and $\boldsymbol{D}_h$, respectively. For boundaries on different line systems and with another sign convention $\boldsymbol{\mu}_h = -\boldsymbol{\lambda}_h^{-1}$ we obtain

$$\boldsymbol{\delta}_e = \boldsymbol{T}_h^t \bar{\boldsymbol{D}} \boldsymbol{T}_e = \boldsymbol{\lambda}_e. \tag{11}$$

This equation is identical to [1, (79)] and constitutes an analytical proof of this formula so far only proved by numerical evidence.

III. Transformation Of The System Matrices To Principal Axes

A. One-Dimensional Discretization

The aim of this section is to diagonalize $\bar{\boldsymbol{Y}}$, $\bar{\boldsymbol{Z}}$, and $\bar{\boldsymbol{V}}$ in the transfer relation from one interface plane to another [1, (30)], which is needed for multilayered structures (see Fig. 1). To this end the tridiagonal system matrices $\bar{\boldsymbol{y}}_1$ and $\bar{\boldsymbol{y}}_2$ are diagonalized beforehand.

First, we examine a single layer in Fig. 1. Our starting point is a hybrid formulation composed of TE_z / TM_z modes. The tangential field components at the layer interfaces A and B are related by [1, (27)–(29)]

$$\begin{bmatrix} \bar{\boldsymbol{H}}_A \\ \bar{\boldsymbol{H}}_B \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{y}}_1 & \bar{\boldsymbol{y}}_2 \\ \bar{\boldsymbol{y}}_2 & \bar{\boldsymbol{y}}_1 \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{E}}_A \\ -\bar{\boldsymbol{E}}_B \end{bmatrix} \tag{12}$$

with

$$\bar{\boldsymbol{H}}_A = \eta_0 \begin{bmatrix} -j\bar{\boldsymbol{H}}_{zA} \\ \bar{\boldsymbol{H}}_{xA} \end{bmatrix} \;;\; \bar{\boldsymbol{E}}_A = \begin{bmatrix} \bar{\boldsymbol{E}}_{xA} \\ -j\bar{\boldsymbol{E}}_{zA} \end{bmatrix}, \tag{13}$$

and the submatrices

$$\bar{\boldsymbol{y}}_1 = \bar{\boldsymbol{y}} \begin{bmatrix} \boldsymbol{\gamma}_h & \\ & \boldsymbol{\gamma}_e \end{bmatrix} \;;\; \bar{\boldsymbol{y}}_2 = \bar{\boldsymbol{y}} \begin{bmatrix} \boldsymbol{\alpha}_h & \\ & \boldsymbol{\alpha}_e \end{bmatrix} \tag{14}$$

with

$$\bar{\boldsymbol{y}} = \begin{bmatrix} -\varepsilon_d \boldsymbol{I}_h & \sqrt{\varepsilon_{re}}\bar{\boldsymbol{\delta}} \\ \sqrt{\varepsilon_{re}}\bar{\boldsymbol{\delta}}^t & \bar{\boldsymbol{\lambda}}_e^2 - \varepsilon_r \boldsymbol{I}_e \end{bmatrix}. \tag{15}$$

According to [1, Sections 2.3–2.4], we use the normalized transformed difference operators $\bar{\boldsymbol{\delta}}$ and $\bar{\boldsymbol{\lambda}}^2$ for the first and second derivative in x direction, respectively. We also need the diagonal matrix for the propagation in y direction

$$\boldsymbol{k}_{\bar{y}}^2 = \bar{\boldsymbol{\lambda}}^2 - \underbrace{(\varepsilon_r - \varepsilon_{re})}_{\varepsilon_d} \boldsymbol{I} \tag{16}$$

and the diagonal matrices

$$\boldsymbol{\alpha} = (\boldsymbol{k}_{\bar{y}} \sinh \boldsymbol{k}_{\bar{y}} \bar{d})^{-1} \;;\; \boldsymbol{\gamma} = (\boldsymbol{k}_{\bar{y}} \tanh \boldsymbol{k}_{\bar{y}} \bar{d})^{-1} \tag{17}$$

with the normalized layer thickness $\bar{d} = k_0 d$.

Because $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are diagonal, the eigenvalues of the matrices $\bar{\boldsymbol{y}}_1$ and $\bar{\boldsymbol{y}}_2$ can be determined from the eigenvalues of the matrix $\bar{\boldsymbol{y}}$. Combination of the eigenvalues and the normalized eigenvectors yields

$$\begin{aligned} \bar{\bar{\boldsymbol{y}}} &= \begin{bmatrix} \boldsymbol{k}_{\bar{y}h}^2 & \\ & -\varepsilon_r \boldsymbol{I}_e \end{bmatrix} = \boldsymbol{X}^t \bar{\boldsymbol{y}} \boldsymbol{X} \\ \boldsymbol{X} &= \begin{bmatrix} \sqrt{\varepsilon_{re}}\boldsymbol{I}_h & -\bar{\boldsymbol{\delta}} \\ \bar{\boldsymbol{\delta}}^t & \sqrt{\varepsilon_{re}}\boldsymbol{I}_e \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} (\varepsilon_{re}\boldsymbol{I}_h + \bar{\boldsymbol{\lambda}}_h^2)^{-\frac{1}{2}} & \\ & (\varepsilon_{re}\boldsymbol{I}_e + \bar{\boldsymbol{\lambda}}_e^2)^{-\frac{1}{2}} \end{bmatrix}. \end{aligned} \tag{18}$$

The eigenvector matrix $\boldsymbol{X}$ is orthonormal, i.e., $\boldsymbol{X}^{-1} = \boldsymbol{X}^t$. We transform the field components by $\boldsymbol{X}$

$$\bar{\bar{\boldsymbol{E}}}_{A,B} = \boldsymbol{X} \bar{\bar{\boldsymbol{E}}}_{A,B} \;;\; \bar{\bar{\boldsymbol{H}}}_{A,B} = \boldsymbol{X} \bar{\bar{\boldsymbol{H}}}_{A,B} \tag{19}$$

and obtain

$$\begin{bmatrix} \bar{\bar{\boldsymbol{H}}}_A \\ \bar{\bar{\boldsymbol{H}}}_B \end{bmatrix} = \begin{bmatrix} \bar{\bar{\boldsymbol{y}}}_1 & \bar{\bar{\boldsymbol{y}}}_2 \\ \bar{\bar{\boldsymbol{y}}}_2 & \bar{\bar{\boldsymbol{y}}}_1 \end{bmatrix} \begin{bmatrix} \bar{\bar{\boldsymbol{E}}}_A \\ -\bar{\bar{\boldsymbol{E}}}_B \end{bmatrix} \tag{20}$$

with the diagonal system matrices

$$\bar{\bar{\boldsymbol{y}}}_1 = \begin{bmatrix} \boldsymbol{k}_{\bar{y}h}^2 \boldsymbol{\gamma}_h & \\ & -\varepsilon_r \boldsymbol{\gamma}_e \end{bmatrix} \;;\; \bar{\bar{\boldsymbol{y}}}_2 = \begin{bmatrix} \boldsymbol{k}_{\bar{y}h}^2 \boldsymbol{\alpha}_h & \\ & -\varepsilon_r \boldsymbol{\alpha}_e \end{bmatrix}. \tag{21}$$

Equation (19) decouples the TE_z / TM_z modes into TE_y / TM_y modes (20). We have used a slightly different formulation than in the immittance approach [5] in the spectral-domain

method in order to obtain real matrices only, which is more beautiful and more convenient for practical calculations. However, a description corresponding to the immittance approach is readily derived.

In order to transfer the tangential field components from interface A to B, we convert (20) to

$$\begin{bmatrix} \bar{\bar{E}}_B \\ \bar{\bar{H}}_B \end{bmatrix} = \begin{bmatrix} \bar{\bar{V}} & \bar{\bar{Z}} \\ \bar{\bar{Y}} & \bar{\bar{V}} \end{bmatrix} \begin{bmatrix} \bar{\bar{E}}_A & \\ & \bar{\bar{H}}_A \end{bmatrix} \tag{22}$$

according to [1, (30)] with the diagonal matrices $\bar{\bar{V}}$, $\bar{\bar{Y}}$, $\bar{\bar{Z}}$ given by $\bar{\bar{V}} = \boldsymbol{X}^t \bar{\boldsymbol{V}} \boldsymbol{X}$ etc.,

$$\bar{\bar{V}} = \bar{\bar{y}}_2^{-1} \bar{\bar{y}}_1 = \begin{bmatrix} \gamma_h \boldsymbol{\alpha}_h^{-1} & \\ & \gamma_e \boldsymbol{\alpha}_e^{-1} \end{bmatrix} = \begin{bmatrix} \cosh \boldsymbol{k}_{\bar{y}h} \bar{d} & \\ & \cosh \boldsymbol{k}_{\bar{y}e} \bar{d} \end{bmatrix} \tag{23}$$

$$\bar{\bar{Z}} = -\bar{\bar{y}}_2^{-1} = -\begin{bmatrix} \boldsymbol{k}_{\bar{y}h}^{-2} \boldsymbol{\alpha}_h^{-1} & \\ & -\varepsilon_r^{-1} \boldsymbol{\alpha}_e^{-1} \end{bmatrix} \tag{24}$$

$$\bar{\bar{Y}} = \bar{\bar{y}}_2 - \bar{\bar{y}}_1 \bar{\bar{y}}_2^{-1} \bar{\bar{y}}_1 = \begin{bmatrix} -\boldsymbol{\alpha}_h^{-1} & \\ & \varepsilon_r \boldsymbol{k}_{\bar{y}e}^{-2} \boldsymbol{\alpha}_e^{-1} \end{bmatrix}. \tag{25}$$

So far we only looked at the layer with ε_{rk} in Fig. 1. As $\boldsymbol{X}$ does not depend on ε_r, this transformation can also be applied to any other homogeneous layer. Consequently we can also transform the relation of the field components of a single layer interface k counted from the bottom of the structure, which is given by [1, (41)]:

$$\bar{\bar{H}}_k = \bar{\bar{Y}}_t^{(k)} \bar{\bar{E}}_k \tag{26}$$

and obtain the recurrence relation

$$\begin{aligned} \bar{\bar{Y}}_t^{(k)} &= (\bar{\bar{Y}}_k + \bar{\bar{V}}_k \bar{\bar{Y}}_t^{(k-1)})(\bar{\bar{V}}_k + \bar{\bar{Z}}_k \bar{\bar{Y}}_t^{(k-1)})^{-1} \\ \bar{\bar{Y}}_t^{(k)} &= \boldsymbol{Y}_{0k} (\bar{\bar{Y}}_t^{(k-1)} + \boldsymbol{Y}_{0k} \boldsymbol{R}_k)(\boldsymbol{Y}_{0k} + \bar{\bar{Y}}_t^{(k-1)} \boldsymbol{R}_k)^{-1}, \end{aligned} \tag{27}$$

with

$$\begin{aligned} \boldsymbol{Y}_{0k} &= \operatorname{diag}(-\boldsymbol{k}_{\bar{y}h}, \varepsilon_r \boldsymbol{k}_{\bar{y}e}^{-1}) \\ \boldsymbol{R}_k &= \operatorname{diag}(\tanh \boldsymbol{k}_{\bar{y}h} \bar{d}_k, \tanh \boldsymbol{k}_{\bar{y}e} \bar{d}_k). \end{aligned} \tag{28}$$

To demonstrate that the result (27) corresponds to that of the immittance approach [5] in the spectral-domain method, we formally replace

$$\begin{aligned} \bar{\bar{Y}}_t^{(k-1)} &\to Y_3^e; \bar{\bar{Y}}_t^{(k)} \to Y_{2L}^e \\ \boldsymbol{Y}_{0k} &\to Y_{TM2}; \boldsymbol{k}_{\bar{y}e} \bar{d}_k \to \gamma_2 t \end{aligned} \tag{29}$$

and obtain

$$Y_{2L}^e = Y_{TM2} \frac{Y_{TM2} + Y_3^e \coth \gamma_2 t}{Y_3^e + Y_{TM2} \coth \gamma_2 t}, \tag{30}$$

which is exactly the same as equation (27) in [5]. Note however that the TE-component is also included in (27).

B. Two-Dimensional Discretization

For two-dimensional discretization we define $\bar{\boldsymbol{y}}_1$ and $\bar{\boldsymbol{y}}_2$ by (14) using the appropriate matrix $\bar{\boldsymbol{y}}$

$$\bar{\boldsymbol{y}} = \begin{bmatrix} \hat{\bar{\boldsymbol{\lambda}}}_{zh}^2 - \varepsilon_r \hat{\boldsymbol{\lambda}}_h & -\hat{\bar{\boldsymbol{\delta}}}_x \hat{\bar{\boldsymbol{\delta}}}_z \\ -\hat{\bar{\boldsymbol{\delta}}}_z^t \hat{\bar{\boldsymbol{\delta}}}_x^t & \hat{\bar{\boldsymbol{\lambda}}}_{xe}^2 - \varepsilon_r \hat{\boldsymbol{\lambda}}_e \end{bmatrix} \tag{31}$$

derived from [1, (190)], also replacing the diagonal matrix for the propagation in y direction $\boldsymbol{k}_{\bar{y}}^2$ by

$$\hat{\boldsymbol{k}}_{\bar{y}}^2 = \hat{\bar{\boldsymbol{\lambda}}}_x^2 + \hat{\bar{\boldsymbol{\lambda}}}_z^2 - \varepsilon_r \hat{\boldsymbol{I}}. \tag{32}$$

All matrices with circumflexes ($\hat{}$) and subscripts x or z are constructed as $\hat{\boldsymbol{D}}_x$ and $\hat{\boldsymbol{D}}_z$, respectively, in [1, (177f)].

The eigenvalue matrix $\bar{\bar{\boldsymbol{y}}}$ is formally the same as in the one-dimensional case (18), but the corresponding eigenvector matrix runs

$$\boldsymbol{X} = \begin{bmatrix} \hat{\bar{\boldsymbol{\lambda}}}_{zh} & -\boldsymbol{J}_z \otimes \bar{\boldsymbol{\delta}}_x \\ \boldsymbol{J}_z^t \otimes \bar{\boldsymbol{\delta}}_x^t & \hat{\bar{\boldsymbol{\lambda}}}_{xe} \end{bmatrix} \begin{bmatrix} (\hat{\bar{\boldsymbol{\lambda}}}_{xh}^2 + \hat{\bar{\boldsymbol{\lambda}}}_{zh}^2)^{-\frac{1}{2}} & \\ & (\hat{\bar{\boldsymbol{\lambda}}}_{xe}^2 + \hat{\bar{\boldsymbol{\lambda}}}_{ze}^2)^{-\frac{1}{2}} \end{bmatrix}. \tag{33}$$

Here $\otimes$ denotes the Kronecker product and $\boldsymbol{J}_z$ a quasi-identity of the same structure as $\bar{\boldsymbol{\delta}}_z$, i.e., $\bar{\boldsymbol{\delta}}_z = \boldsymbol{J}_z \bar{\boldsymbol{\lambda}}_{ze}$.

IV. Conclusion

The matrix analysis of the method of lines has been improved. A general procedure for the computation of the eigenvalues of the difference operators and the transformation matrices and new formulas for diagonalized matrices for the transfer of the field components from one interface to the other are obtained. Both increase the mathematical clarity and save numerical effort.

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